Semigroup representations (Spring 2016)

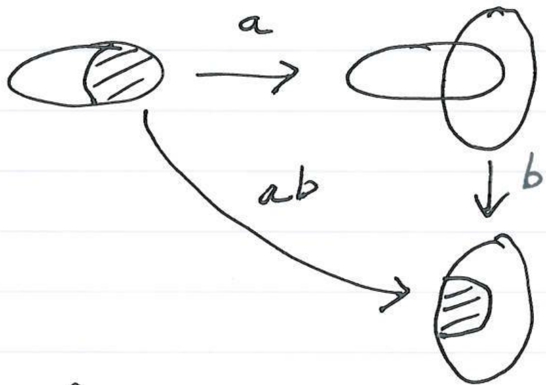
1. Semigroup basics

- S = finite semigp.
- (three running) Eg's: $[n] = \{1, \dots, n\}$

(i). S_n = all bijections $[n] \xrightarrow{a} [n]$ under composition.

(ii). I_n = all partial bijections $X \xrightarrow{a} Y$, $X, Y \subseteq [n]$

under composition:



(all fcns, actions, etc on the right)

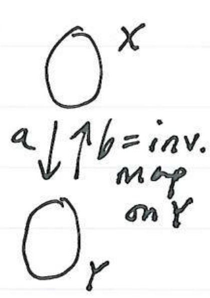
(ii). T_n = all maps $[n] \xrightarrow{a} [n]$ under composition.

- inverses in semi-gps: an inverse of $a \in S$ is a b s.t

$$(*) \quad \begin{aligned} aba &= a \\ bab &= b \end{aligned}$$

(i). S_n a monoid s.t $\forall a\, \exists!\, b$ with $ab = 1 = ba$ ($\not\Rightarrow (*)$)

i.e: a gp!



(ii). I_n a monoid s.t. $\forall a\, \exists!\, b$ satisfying $(*)$

($\Rightarrow$ $ab = id_X$ idempotents)

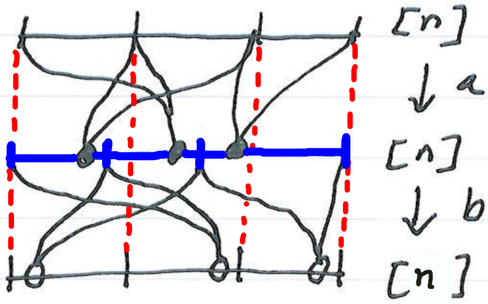
$ba = id_Y$

i.e: I_n an inverse monoid

(iii). T_n a monoid s.t $\forall a\, \exists$ (many) b satisfying $(*)$

fibres of a (= equiv. classes of ker a)

e.g.:



T_n a <u>regular monoid</u>

from now on S = <u>finite regular monoid</u>

- <u>Structure</u> : Green's relations in S_n, I_n, T_n

1. $a \mathcal{L} b \iff im(a) = im(b)$

2. $a \mathcal{R} b \iff$ fibres of a = fibres of b

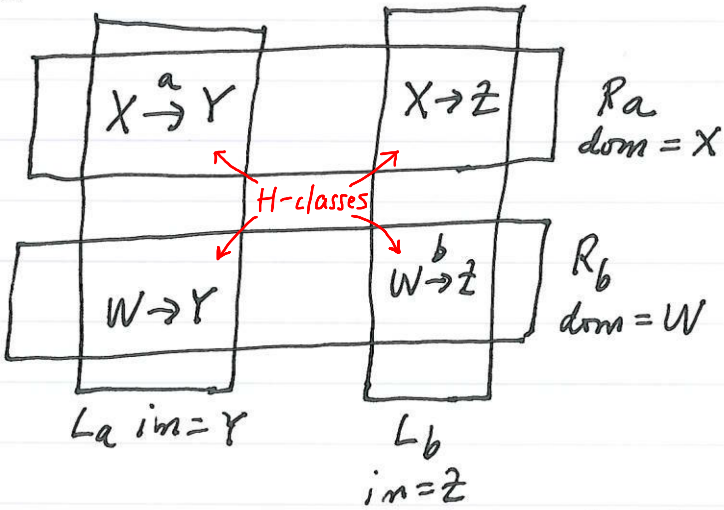
   $\left(\overset{S_n}{\underset{I_n}{\iff}} dom(a) = dom(b) \right)$

3. $a \mathcal{J} b \iff |im(a)| = |im(b)|$

4. $a \mathcal{H} b \iff 1+2.$

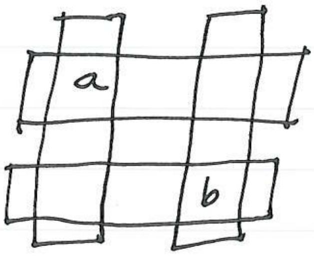
S_n: these are all trivial! (any a, b are related)

I_n:
(by 1, 2)



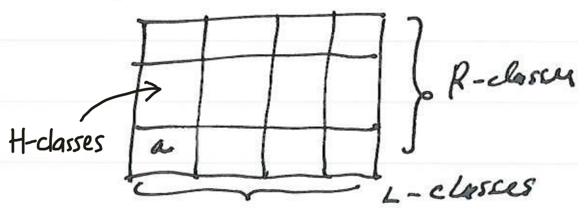
R and L commute to give nice "eggbox" picture

by 3: $a \mathcal{J} b \iff$



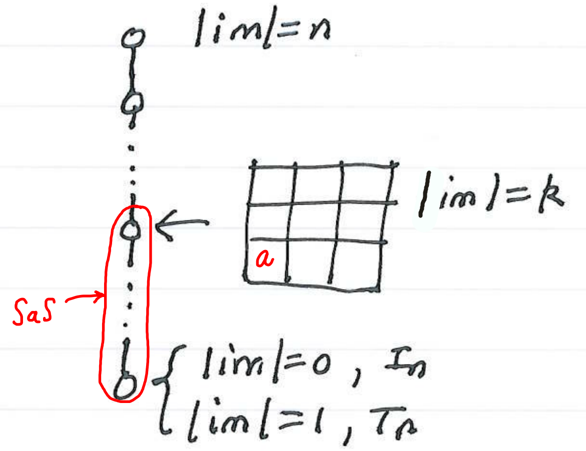
belong in same eggbox

i.e.: J_a = J-class of a :



These pictures hold for all S (finite regular monoids).

In I_n / T_n the $\mathcal{J}$-classes naturally ordered:



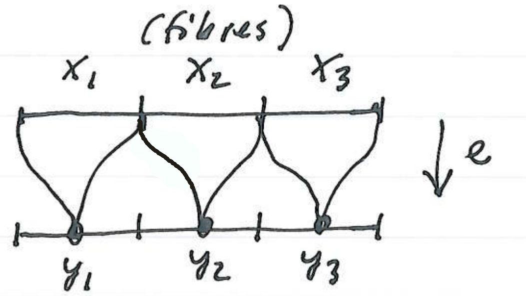
In general, $J_a \le J_b$ $\overset{\text{def}}{\iff}$ $SaS \subseteq SbS$ partial order.

- <u>Idempotents / subgroups</u>: idempotent $e = e^2$

I_n: $e: X \xrightarrow{id_X} X$, $X \subseteq [n]$

Fix domain X: only one such map $\Rightarrow$ every $\mathcal{R}$-class has exactly one idempotent (similarly every $\mathcal{L}$-class)

T_n:

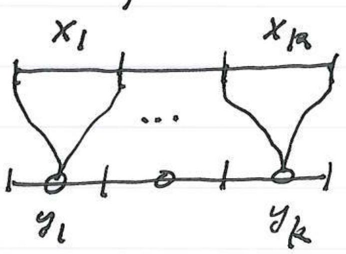


$e: X_i \mapsto y_i$ with $y_i \in X_i$

Fix the fibres and wiggle the y_i inside them (or conversely) $\Rightarrow$ every $\mathcal{R}$-class has at least one idem. (similarly $\mathcal{L}$-classes).

Similarly for every inverse or regular semigp

$\mathcal{H}$-class of in T_n



contains all bijections $\{X_1, \dots, X_k\} \to \{y_1, \dots, y_k\}$

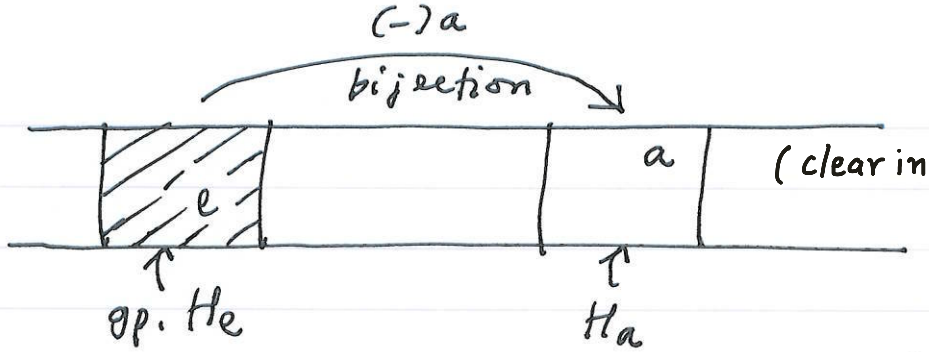
i.e.: a copy of S_k in T_n

In general H_e a <u>subgp.</u> of S (with identity e)

If G a subgroup of S then $G \subseteq H_e$ for some e.

(hence the H_e are <u>maximal</u> subgroups)

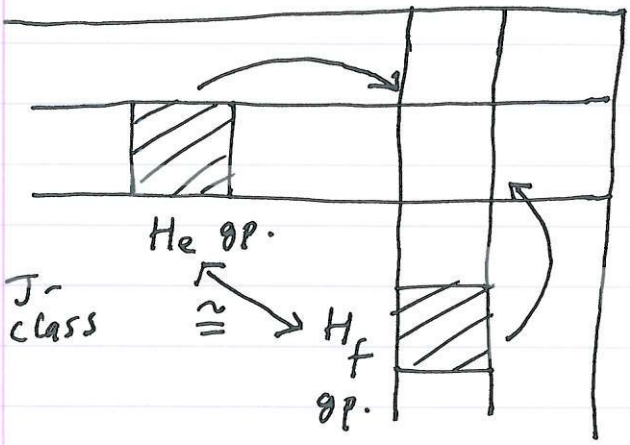
R-class:



(clear in I_n)

$\Rightarrow$ every element of H_a has unique expression ga ($g \in H_e$)

(and similarly in an L-class).



Two gp. H-classes in a J-class isomorphic.